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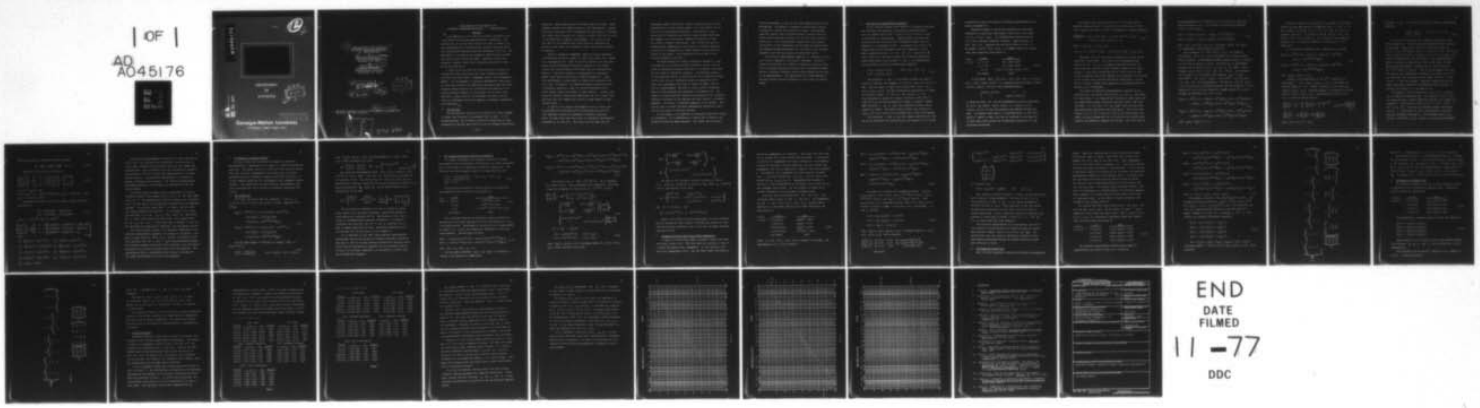

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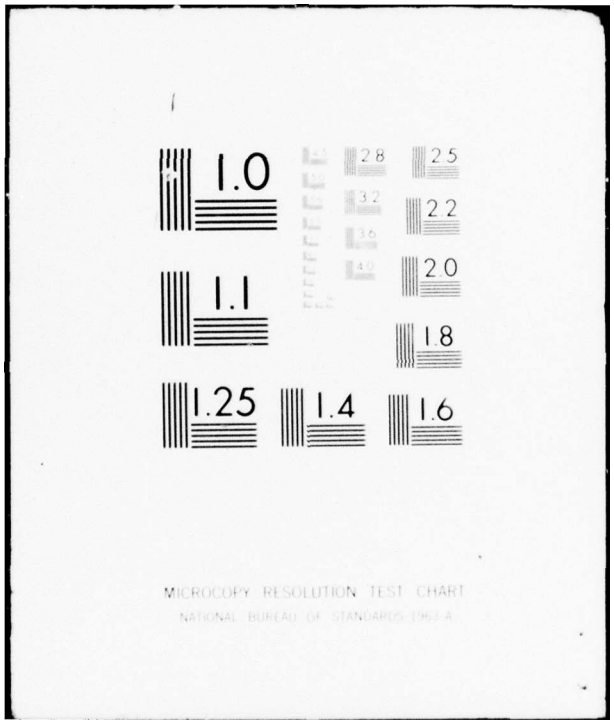
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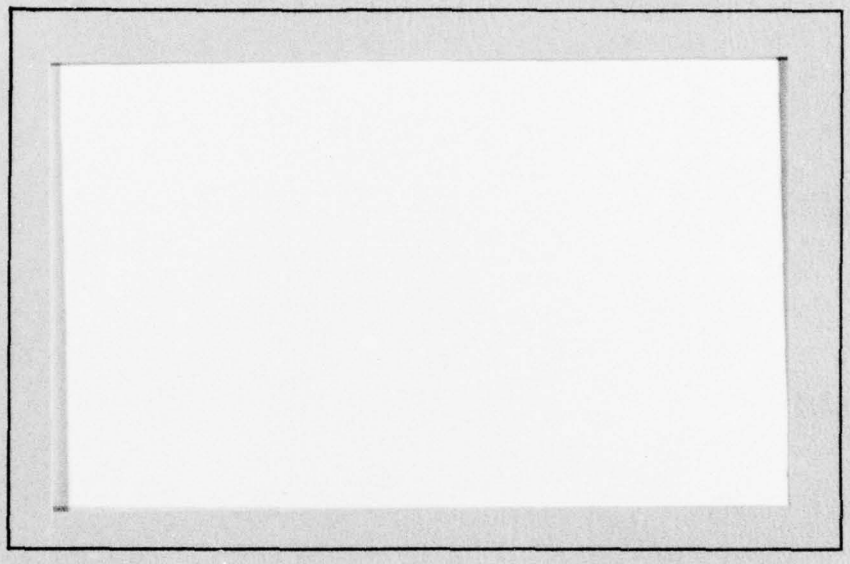

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A NEW APPROACH TO THE ANALYSIS OF STOCHASTIC LANCHESTER PROCESSES. I. TIME EVOLUTION.

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A NEW APPROACH TO THE ANALYSIS OF STOCHASTIC LANCHESTER PROCESSES: I. TIME EVOLUTION

Abstract

A new approach to the study of stochastic Lanchester processes based on diffusion approximations is presented. The distribution of the two force levels over time is shown to be well approximated by a nonstationary bivariate Gaussian diffusion process with specified mean and covariance structure. The approximation is based on an asymptotic analysis which assumes the initial force levels are large. Numerical studies are presented, however, which show surprising accuracy for force levels as small as 30.

A wide variety of attrition structures are discussed including the linear and square law cases, Helmbold's general attrition structure, Karr's engagement model, and heterogeneous models. The development of tractable mathematical expressions for the time evolution of complicated Lanchester-type attrition processes makes possible the introduction and analysis of decision theoretic aspects to the problem such as force level decisions, combat tactics, reinforcement decisions, and the value of information about the opponent's strengths, weaknesses, and strategies.

1. Introduction

The mathematical theory of the attrition of forces engaged in combat was initiated by Lanchester [12] in 1914. In his pioneering work, the attrition process was formulated as being deterministic and was described by a set of ordinary differential

equations. Many advances have been made beyond this work. These advances include such complications as time dependencies, complex attrition structures, force inhomogeneities, and aspects of control. The reader should consult the tutorial by Taylor [14] for a survey of this work and a complete bibliography. It is fair to say that most of the advances have come within the deterministic point of view, while the stochastic theory has been much slower to develop.

Within a stochastic framework, some of the more important contributions are due to Clark [4], Karr [10], Weiss [15], Brooks [3], and Helmbold [9]. Both [15] and [3] present the idea that under certain conditions a stochastic analysis is not needed, because the results differ little from that of a deterministic analysis. Karr [10] presents a methodology for developing Markov chain models from explicit combat assumptions. Using this methodology, a large number of models are presented; however, none is analyzed. An immediate problem encountered with a stochastic analysis is that the interesting models which arise usually cannot be solved, at least exactly. In fact, only Clark [4] presents closed form expressions for the "linear" law, and these are very complex and involve a large number of exponential terms.

In spite of the lack of development of such a theory and the mathematical intractabilities encountered, there are two very important reasons for adopting a stochastic approach. First, we take issue with the notion of "stochastic determinism" presented in [3] and [15]. The claim has been made that for

reasonably large initial force levels, quantities such as the probability of victory become nearly 0 or 1 and reflect the results of the deterministic analysis. Nevertheless, other quantities are important and exhibit large random fluxuations. For example, the force levels themselves plotted over time will exhibit enormous variability. This variability must be characterized, especially if one wishes to introduce elements of tactics or strategies which might depend on the actual evolution of the combat. This variability can be seen in the results presented in section 4.

The second reason for using a stochastic analysis is that it allows one to introduce decision theoretic concepts to the study of Lanchester processes and hence to introduce a new role for this area of operations research. If Lanchester theory is to have practical value, it should allow one to address problems such as the value of information about the opponent's force levels and strategies, what type of battle to fight, when to call in reinforcements, and when to avoid conflict altogether. Some of these questions are addressed in Schreiber [13]. The answers to these questions require the introduction of a decision theoretic structure to the problem, and they require a stochastic approach. Of course, a stochastic approach is not enough. One must be able to obtain tractable mathematical expressions which can then be used in a decision analysis.

In this paper, a new approach to stochastic Lanchester theory is introduced. It is approximate in nature and is based on the method of diffusion approximations. The reader should consult

Gaver and Lehoczky [5] and [6] for other applications of this methodology. The method is asymptotic and provides results assuming the initial force levels are large. Nevertheless, numerical results will be presented which indicate that the method offers excellent accuracy for initial force levels as small as 30. This accuracy when coupled with the analytic tractability of the expressions derived make the method ideal for studying random fluxuations as a function of the relevant parameters and tactical aspects of Lanchester theory.

The diffusion approximation approach is introduced in section 2 and is applied to the "square" law model. Section 3 contains an analysis of a number of different attrition structures including nonlinear structures and heterogeneous models. Section 4 presents numerical results which assess the accuracy of the approximation. The application of this methodology to solving tactical decision problems is the subject of another report.

2. The Diffusion Approximation Approach

In this section, Markov chain models of Lanchester processes are introduced and analyzed using diffusion approximations. Only the simple "square law" studied in this section in order to illustrate the methodology. Complex models including non-linear attrition structures and heterogeneous models will be examined in a subsequent section. Throughout this section, we assume there are two opponents each with homogeneous forces which undergo attrition. If $x(t)$ and $y(t)$ denote the two force levels as a function of time, then classical Lanchester theory would model the functions x and y as deterministic and satisfying a system of differential equations

$$\begin{aligned} x'(t) &= -f(x(t), y(t)) \\ y'(t) &= -g(x(t), y(t)). \end{aligned} \qquad , \; x(0) = x_0, \;\; y(0) = y_0 \tag{2.1}$$

The functions f and g are chosen to model the type of conflict involved. For example, if $f(x, y) = axy$ and $g(x, y) = bxy$, then the model is called the "linear law" (since $b(x_0 - x(t)) = a(y_0 - y(t))$) and was introduced by Lanchester as a model of "ancient" combat. If $f(x, y) = ay$ and $g(x, y) = bx$, then the model is called the "square law"(since $b(x_0^2 - x^2(t)) = a(y_0^2 - y^2(t))$) and was introduced as a model of "modern" combat. Many functions f and g have been proposed to model various aspects of combat. The reader should consult the tutorial by Taylor [14] for a survey of such models and a large bibliography.

The functions f and g can also depend explicitly on time. We do not allow for this possibility in this paper, but it should

be pointed out that the method of diffusion approximations can easily accomodate it.

Stochastic models of attrition processes are typically formulated as continuous time Markov chains with state space $S = \{(i, j) | i \geq 0, j \geq 0\}$. The deterministic system given by (2.1) has a natural stochastic analogue. Specifically, let $X(t)$ and $Y(t)$ represent the two force levels at time t. We assume $\{(X(t), Y(t)), t \geq 0\}$ is a Markov chain with transitions and transition rates given by

t		$t + dt$	Rate	
(i, j)	$\rightarrow$	$(i-1, j)$	$f(i, j)dt + o(dt)$	
		$(i, j-1)$	$g(i, j)dt + o(dt)$	(2.2)
		(i, j)	$1-(f(i, j) + g(i, j))dt + o(dt)$	
		all others	$o(dt)$.	

If we define $dX(t) = X(t + dt) - X(t)$ and $dY(t) = Y(t + dt) - Y(t)$, then $(dX(t), dY(t))$ is a random vector with mean $(-f(X(t), Y(t))dt, -g(X(t), Y(t))dt)$ and covariance matrix

$$\begin{pmatrix} f(X(t), Y(t))dt & 0 \\ 0 & g(X(t), Y(t))dt \end{pmatrix}$$

to terms of order dt. For the deterministic process specified by (2.1), the vector $(dx(t), dy(t))$ is $(-f(x(t), y(t)), -g(x(t), y(t)))$, so the two processes are adapted through the mean. The Markov chain model is a generalization of the deterministic model in that (2.1) can be recovered in the mean but it also takes into account the fluxuations encountered in real attrition situations.

The Markov chain model specified by (2.2) brings with it immediate complications. To specify the distribution of (X(t), Y(t)) exactly, one needs to solve the Kolmogorov forward equations:

$$\frac{dP(t,m,n)}{dt} = f(m+1, n)P(t, m+1, n) + g(m, n+1)P(t, m, n+1) - (f(m, n) + g(m, n))P(t, m, n), \tag{2.3}$$

where $0 < m \leq x_0$, $0 < n \leq y_0$ and

$$P(t, m, n) = P((X(t), Y(t)) = (m,n) | (X(0), Y(0)) = (x_0, y_0)).$$

Equations (2.3) are essentially impossible to solve in any convenient closed form. Only Clark [4] has developed exact expressions for $P(t, m, n)$; however, they were only for the linear law case, and the results involve a double sum of exponentials. Such exact expressions are of little value in providing information about the behavior of the process as a function of the initial values and the attrition coefficients. If one wishes to adopt a Markov chain approach, an alternate method of analysis must be developed.

The method of diffusion approximations provides such an approach when the initial force levels x_0 and y_0 are large. The state space S is replaced by a continuous state space, and the sample paths become continuous functions rather than step functions. If $X(t)$ and $Y(t)$ are large, then over any short time period many transitions will take place. Since the total change in the X and Y processes will be the sum of many unit steps, the total change will be, by virtue of the central limit theorem, approximately normally distributed. The first step in

the approximation is to replace (2.1) and (2.2) by a diffusion process defined by the Ito stochastic differential equations (s.d.e. in the sequel)

$$\begin{aligned} dX(t) &= -f(X(t), Y(t))dt + (f(X(t), Y(t)))^{1/2}dW_1(t) \\ dY(t) &= -g(X(t), Y(t))dt + (g(X(t), Y(t)))^{1/2}dW_2(t) \end{aligned} \tag{2.4}$$

with $X(0) = x_0$, $Y(0) = y_0$ a.s., and where $W_1(t)$ and $W_2(t)$ are independent standard Wiener processes.

The increments of $\{(X(t), Y(t)), t \geq 0\}$ are normally distributed. The s.d.e. (2.4) corresponds to a partial differential equation, the Kolmogorov forward equation, which is similar to (2.3) and no easier to solve. Nevertheless, an approximate analysis can be carried out if x_0 and y_0 are large. Specifically, we let $X(0) = Nu(0)$ and $Y(0) = Nv(0)$. We introduce, in the spirit of the central limit theorem, $U_N(t) = (X(t) - Nu(t))/N^{1/2}$ and $V_N(t) = (Y_N(t) - Nv(t))/N^{1/2}$. The functions $u(t)$ and $v(t)$ are deterministic. The process $\{(U_n(t), V_N(t)), t \geq 0\}$ will also be a diffusion process, and the system of s.d.e.'s governing it can be found using Ito's Lemma (Arnold [1], p.90 or Gihkman and Skorokhod [7], p.27) to be

$$\begin{aligned} dU_N(t) &= -N^{1/2}(u'(t) + f(Nu(t) + N^{1/2}U_N(t), Nv(t) + N^{1/2}V_N(t)))dt \\ &\quad + (f(Nu(t) + N^{1/2}U_N(t), Nv(t) + N^{1/2}V_N(t)))^{1/2}dW_1(t) \\ dV_N(t) &= -N^{1/2}(v'(t) + g(Nu(t) + N^{1/2}U_N(t), Nv(t) + N^{1/2}V_N(t)))dt \\ &\quad + (g(Nu(t) + N^{1/2}U_N(t), Nv(t) + N^{1/2}V_N(t)))^{1/2}dW_2(t) \end{aligned} \tag{2.5}$$

with $U_N(0) = V_N(0) = 0$ a.s.

We wish to study (2.5) assuming $N \to \infty$; however, to proceed further we must make specific assumptions about f and g. In this section, we assume $f(x, y) = ay$ and $g(x, y) = bx$, the quadratic law case. In general we assume $f(X(t), Y(t))$ and $g(X(t), Y(t))$ are of order N. This allows for an asymptotic analysis of (2.5) based on Theorem K of Barbour [2] or on Kurtz [11].

In the specific quadratic case, equations (2.5) become

$$\begin{aligned} dU_N(t) &= -N^{1/2}(u'(t) + av(t))dt - aY_N(t)dt + O(N^{-1/2})dt \\ &\quad + (av(t) + O(N^{-1/2}))^{1/2}dW_1(t) \\ dV_N(t) &= -N^{1/2}(v'(t) + bu(t))dt - bX_N(t)dt + O(N^{-1/2})dt \\ &\quad + (bu(t) + O(N^{-1/2}))^{1/2}dW_2(t) \end{aligned} \tag{2.6}$$

with $U_N(0) = V_N(0) = 0$ a.s.

Clearly, the coefficients of the $N^{1/2}$ terms must be identically 0 for $\{(U_N(t), V_N(t)), t \geq 0\}$ to converge to a sensible limit as $t \to \infty$. Indeed, Theorem K of [2] shows that $\{(U_N(t), V_N(t)), t \geq 0\}$ converges weakly to a limiting diffusion process $\{(U(t), V(t)), t \geq 0\}$ satisfying the system of Ito s.d.e.'s given in matrix form by

$$\begin{pmatrix} dU(t) \\ dV(t) \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix} \begin{pmatrix} U(t) \\ V(t) \end{pmatrix} dt + \begin{pmatrix} (av(t))^{1/2} & 0 \\ 0 & (bu(t))^{1/2} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

or

$$\begin{pmatrix} dU(t) \\ dV(t) \end{pmatrix} = \underset{\sim}{A}_t \begin{pmatrix} U(t) \\ V(t) \end{pmatrix} dt + \underset{\sim}{B}_t \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} \tag{2.7}$$

with $U(0) = V(0) = 0$ a.s.,

provided u and v satisfy the system of ordinary differential equations

$$\begin{aligned} u'(t) &= -av(t) \qquad u(0) = x_0, \quad v(0) = y_0 \\ v'(t) &= -bu(t) \end{aligned} \tag{2.8}$$

Of course (2.8) are just the ordinary deterministic Lanchester equations. The diffusion approximation method consists of replacing $(X(t), Y(t))$ which is identically equal to $N(u(t), v(t)) + N^{1/2}(U_N(t), V_N(t))$ by $N(u(t), v(t)) + N^{1/2}(U(t), V(t))$, where the latter is given by (2.7) and (2.8) and is far easier to characterize then the former. This approximation is based on exactly the same idea as the central limit theorem which would approximate a binomial (N, p) distribution by $Np + N^{1/2}$ Normal $(0, p(1-p))$. The first two moments have been fit exactly, and probability statements are exactly correct in the limit and very accurate for even samll values of N provided p is not too extreme. Numerical comparisons will be given later which show that this method provides excellent a accuracy for values of N as small as 30.

The method is useful because the $(U(t), V(t))$ process can be explicitly characterized. The system of s.d.e.'s (2.7) is linear, and much is known about such systems (for example [1], chapter 8). Specifically, $E(U(t), V(t)) = (0,0)$ for all t and $(U(t), V(t))$ will have a bivariate normal distribution for all t. The process is thus a nonstationary, bivariate, Gaussian diffusion with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}_t = E((U(t), V(t))^T(U(t), V(t)))$ which is the unique non-negative

definite solution of the matrix Ricatti equation

$$\dot{\underset{\sim}{\Sigma}}_t = \underset{\sim}{A}_t\underset{\sim}{\Sigma}_t + \underset{\sim}{\Sigma}_t\underset{\sim}{A}_t^T + \underset{\sim}{B}_t\underset{\sim}{B}_t^T, \quad \underset{\sim}{\Sigma}_0 = \underset{\sim}{0} \tag{2.9}$$

Equations (2.9) can also be written

$$\begin{pmatrix} \dot{\sigma}_{11}(t) \\ \dot{\sigma}_{12}(t) \\ \dot{\sigma}_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & -2a & 0 \\ -b & 0 & -a \\ 0 & -2b & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11}(t) \\ \sigma_{12}(t) \\ \sigma_{22}(t) \end{pmatrix} + \begin{pmatrix} av(t) \\ 0 \\ bu(t) \end{pmatrix} \tag{2.10}$$

or $\underset{\sim}{\dot{\sigma}}(t) = \underset{\sim}{C}_t\underset{\sim}{\sigma}(t) + \underset{\sim}{D}_t$

with $\sigma_{ij}(0) = 0$, $\sigma_{11}(t) = \mathrm{Var}(U(t))$, $\sigma_{22}(t) = \mathrm{Var}(V(t))$, and $\sigma_{12}(t) = \mathrm{Cov}(U(t), V(t))$.

Equations (2.8) and (2.10) can be easily solved, and their solutions are

$$\begin{aligned} u(t) &= x_0\cosh(kt) - cy_0\sinh(kt) \\ v(t) &= y_0\cosh(kt) - c^{-1}x_0\sinh(kt) \end{aligned} \tag{2.11}$$

with $c = (a/b)^{1/2}$ and $k = (ab)^{1/2}$,

$$\begin{pmatrix} \sigma_{11}(t) \\ \sigma_{12}(t) \\ \sigma_{22}(t) \end{pmatrix} = \frac{1}{8k}\begin{pmatrix} a & a & a \\ k & -k & 0 \\ b & b & -b \end{pmatrix}\begin{pmatrix} d_1\exp(kt) + d_2\exp(-kt) + d_3\exp(-2kt) \\ d_4\exp(2kt) + d_5\exp(kt) + d_6\exp(-kt) \\ d_7\exp(kt) + d_8\exp(-kt) + d_9 \end{pmatrix} \tag{2.12}$$

with

$d_1 = \frac{1}{3}(y_0(1-c) + x_0(1-c^{-1}))$, $d_2 = y_0(1+c) + x_0(1+c^{-1})$,

$d_3 = -\frac{1}{3}(y_0(4-2c) + x_0(4-2c^{-1}))$, $d_4 = \frac{1}{3}(y_0(4-2c) + x_0(4-2c^{-1}))$,

$d_5 = -(y_0(1-c) + x_0(1-c^{-1}))$, $d_6 = -\frac{1}{3}(y_0(1+c) + x_0(1+c^{-1}))$,

$d_7 = 2(y_0(1+c) - x_0(1+c^{-1}))$, $d_8 = -2(y_0(1-c) - x_0(1-c^{-1}))$,

$d_9 = -4(cy_0 - c^{-1}x_0)$.

The diffusion approximation thus gives the time evolution of $(X(t), Y(t))$ as the sum of the deterministic process given by (2.11) and a stochastic noise process which has been completely characterized. This illustrates that the deterministic approach represented by (2.1) is a first-order approximation to the Lanchester process. A second-order approximation is obtained by adding a stochastic noise process. The diffusion approximation methodology provides a link between the deterministic and stochastic models.

The reader should be aware that the importance of using a stochastic analysis will depend on the size of N. For very large values of N, the deterministic term will dominate the stochastic term. In such cases, the deterministic approach will be completely adequate. On the other hand, when N is small, say less than 500, wide fluxuations are possible. To illustrate, in a square law case with $a = b = .05$, $x_0 = y_0 = 300$, after 15 time units the x force level could be as low as 102 or as high as 183. If $x_0 = y_0 = 30$, the x force level may be between 1 and 27. The deterministic model gives, therefore, very misleading results.

The reader should be aware of an important limitation of the diffusion approximation approach, the problem of boundaries. One must be aware that the $(X(t), Y(t))$ process is confined to the first quadrant with the x and y axes being absorbing barriers. The bivariate normal distribution will be positive over the entire plane, while the process can never leave the first quadrant. The approximation will be trustworthy only if most of the mass of the normal distribution is in the first quadrant.

3. An Analysis of Specific Models

In this section, we illustrate the method of diffusion approximations by applying it to a number of different Lanchester processes. The models chosen for study have been selected because they have classical significance (the linear law), illustrate mathematical points (Helmbold's general attrition structure), or involve complex modelling (the heterogeneous and engagement models of Karr). The reader will see how diffusion approximations provide a unifying tool for the study of stochastic Lanchester processes.

A. The Linear Law

The linear law arises when the functions f and g in (2.2) are $f(i,j) = aij/N$ and $g(i,j) = bij/N$. Equation (2.6) becomes

$$\begin{aligned} dU_N(t) &= -N^{1/2}(u'(t) + au(t)v(t))dt + O(N^{-1/2})dt \\ &\quad -(av(t)U_N(t) + au(t)V_N(t))dt \\ &\quad + (au(t)v(t) + O(N^{-1/2}))^{1/2}dW_1(t) \\ dV_N(t) &= -N^{1/2}(v'(t) + bu(t)v(t))dt + O(N^{-1/2})dt \\ &\quad -(bv(t)U_N(t) + bu(t)V_N(t))dt \\ &\quad + (bu(t)v(t) + O(N^{-1/2}))^{1/2}dW_2(t) \end{aligned} \tag{3.1}$$

One may again appeal to Theorem k of Barbour (1974) to prove that if

$$\begin{aligned} u'(t) &= -au(t)v(t) \\ v'(t) &= -bu(t)v(t) \end{aligned} \qquad u(0) = X(0)/N, \quad v(0) = Y(0)/N, \tag{3.2}$$

then $\{(U_N(t), V_N(t)), t \geq 0\}$ converges weakly to $\{(U(t), V(t)), t \geq 0\}$ which satisfies (2.7) with

$$\underset{\sim}{A}_t = \begin{pmatrix} -av(t) & -au(t) \\ -bv(t) & -bu(t) \end{pmatrix} \quad \text{and} \quad \underset{\sim}{B}_t = \begin{pmatrix} (au(t)v(t))^{1/2} & 0 \\ 0 & (bu(t)v(t))^{1/2} \end{pmatrix}.$$

The diffusion approximation gives $(X(t), Y(t)) \approx N(u(t), v(t)), + N^{1/2}(U(t), V(t))$ where $u(t)$ and $v(t)$ satisfy the usual Lanchester equations (3.2). The distribution of $(X(t),V(t))$ will thus be approximately bivariate normal with the usual deterministic result as a mean and covariance $N\underset{\sim}{\Sigma}_t$ where $\underset{\sim}{\Sigma}_t$ can be derived from (2.9) or equivalently (2.10) with

$$\underset{\sim}{C}_t = \begin{pmatrix} -2av(t) & -2au(t) & 0 \\ -bv(t) & -(av(t) + bu(t)) & -au(t) \\ 0 & -2bv(t) & -2bu(t) \end{pmatrix}, \quad \underset{\sim}{D}_t = \begin{pmatrix} au(t)v(t) \\ 0 \\ bu(t)v(t) \end{pmatrix} \tag{3.3}$$

The deterministic equations (3.2) can be solved in closed form; however, the covariance equations, having time dependent coefficients, will typically not be. Nevertheless, (3.3) can be solved by straightforward numerical techniques. If numerical techniques are required, then the f and g functions can be made to depend explicitly on time. Alternative expressions for the solution of (3.3) are given in [1], Chapter 8.

It is important to note that using diffusion approximation methodology reduces the problem of studying stochastic Lanchester processes to that of solving ordinary differential equations which arise from the deterministic approach and the covariance matrix. As such, the stochastic approach is in principle no harder than the deterministic approach.

B. The General Attrition Structure of Helmbold

In a paper by Helmbold [8], a general attrition structure was proposed which incorporates inefficiencies of scale associated with a vary large force fighting a relatively smaller one. The attrition structure proposed consists of replacing (2.1) by

$$\begin{aligned} x'(t) &= -ax^c(t)y^{1-c}(t), \qquad x(0) = x_0, \quad y(0) = y_0 \\ y'(t) &= -by^c(t)x^{1-c}(t), \\ &\text{and } 0 < c < 1. \end{aligned} \tag{3.4}$$

The associated stochastic version consists of transitions and transition rates similar to (2.2) given by

t	$t + dt$	Rate
(i,j) $\rightarrow$	$(i-1,j)$	$ai^c j^{1-c}dt + o(dt)$
	$(i,j-1)$	$bj^c i^{1-c}dt + o(dt)$
	(i,j)	$1 - (ai^c j^{1-c} + bj^c i^{1-c})dt + o(dt)$
	all others	$o(dt)$.

(3.5)

The non-linear form of the transition rates precludes any possibility of an exact expression for the distribution of $X(t)$, $Y(t)$) being derived. Nevertheless, the process can be approximated by using s.d.e.'s which are obtained by matching the first and second moments. Specifically, we write

$$\begin{aligned} dX(t) &= -a(X(t))^c(Y(t))^{1-c}dt + (a(X(t))^c(Y(t))^{1-c})^{1/2}dW_1(t) \\ dY(t) &= -b(X(t))^{1-c}(X(t))^c dt + (b(X(t))^{1-c}(Y(t))^c)^{1/2}dW_2(t), \end{aligned} \tag{3.6}$$

with $X(0) = x_0$, $Y(0) = y_0$ a.s.

We once again introduce $U_N(t)$ and $V_N(t)$ as defined in section 2 and using Ito's Lemma write

$$dU_N(t) = -N^{1/2}(u'(t) + aN^{-1/2}(Nu(t) + N^{1/2}U_N(t))^c(Nv(t) + N^{1/2}V_N(t))^{1-c})dt$$
$$+ (aN^{-1}(Nu(t) + N^{1/2}U_N(t))^c(Nv(t) + N^{1/2}V_N(t))^{1-c})^{1/2}dW_1(t)$$

$$dV_N(T) = -N^{1/2}(v'(t) + bN^{-1/2}(Nu(t) + N^{1/2}U_N(t))^{1-c}(Nv(t) + N^{1/2}V_N(t))^c)dt$$
$$+ (bN^{-1}(Nu(t) + N^{1/2}U_N(t))^{1-c}(Nv(t) + N^{1/2}V_N(t))^c)^{1/2}dW_2(t) \tag{3.7}$$

The factors such as $(Nu(t) + N^{1/2}U_N(t))^c$ can be expanded in a power series. Upon performing these expansions, collecting terms, and dropping all terms which are $O(N^{-1/2})$ we find

$$\begin{pmatrix} dU_N(t) \\ dV_N(t) \end{pmatrix} = -N^{1/2}\begin{pmatrix} u'(t) + au^c(t)v^{1-c}(t) \\ v'(t) + bu^{1-c}(t)v^c(t) \end{pmatrix} dt$$
$$- \begin{pmatrix} ca(\frac{v(t)}{u(t)})^{1-c} & a(1-c)(\frac{v(t)}{u(t)})^{-c} \\ (1-c)b(\frac{v(t)}{u(t)})^{c} & cb(\frac{v(t)}{u(t)})^{c-1} \end{pmatrix} \begin{pmatrix} U_N(t) \\ V_N(t) \end{pmatrix} dt \tag{3.8}$$
$$+ \begin{pmatrix} (au^c(t)v^{1-c}(t))^{1/2} & 0 \\ 0 & (bu^{1-c}(t)v^c(t))^{1/2} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}.$$

If u and v satisfy

$$\begin{aligned} u'(t) &= -au^c(t)v^{1-c}(t) & u(0) &= x_0/N \\ v'(t) &= -bu^{1-c}(t)v^c(t) & v(0) &= y_0/N \end{aligned}, \tag{3.9}$$

then $\{(U_N(t), V_N(t)), t \geq 0\}$ converges weakly to $\{(U(t), V(t)), t \geq 0\}$ which satisfies (2.7) with

$$\underset{\sim}{A}_t = \begin{pmatrix} ca\left(\frac{v(t)}{u(t)}\right)^{1-c} & a(1-c)\left(\frac{v(t)}{u(t)}\right)^{-c} \\ (1-c)b\left(\frac{v(t)}{u(t)}\right)^{c} & cb\left(\frac{v(t)}{u(t)}\right)^{c-1} \end{pmatrix} \quad \text{and}$$

$$\underset{\sim}{B}_t = \begin{pmatrix} (au^c(t)v^{1-c}(t))^{1/2} & 0 \\ 0 & (bu^{1-c}(t)v^c(t))^{1/2} \end{pmatrix}.$$

The mean of the stochastic process is given by $N(u(t), v(t))$, while the covariance is given by $N\underset{\sim}{\Sigma}_t$ where $\underset{\sim}{\Sigma}_t$ satisfies (2.9) or equivalently (2.10) with

$$\underset{\sim}{C}_t = -\begin{pmatrix} 2car^{1-c}(t) & 2a(1-c)r^{-c}(t) & 0 \\ (1-c)br^c(t) & c(ar^{1-c}(t) + br^{c-1}(t)) & a(1-c)r^{-c}(t) \\ 0 & 2(1-c)br^c(t) & 2cbr^{c-1}(t) \end{pmatrix}$$

with $r(t) = v(t)/u(t)$ and

$$\underset{\sim}{D}_t = (au^c(t)v^{1-c}(t),\ 0,\ bu^{1-c}(t)v^c(t))^T.$$

Again, (2.10) must be solved numerically, but the stochastic process problem has been reduced to solving two systems of ordinary differential equations, even in the case of highly nonlinear transition rates.

C. Homogeneous Linear Law Process With Engagements

A number of interesting complex stochastic models have been introduced by Karr [10]. The first which we consider is that of a linear law process with engagements. In this process, each side has a homogeneous force. The two sides suffer attrition by

one-on-one engagements or encounters. This means that each side can be thought of as being divided into two parts - a part which is involved in engagements and a part seeking such involvements. To make the stochastic process Markovian, one must assume that engagements last for an exponential length of time with parameter μ. Furthermore, each engagement has three possible outcomes: side i wins with probability p_i, i=1,2 or both survive with probability p_3 $(p_1 + p_2 + p_3 = 1)$. Engagements are assumed to develop at a rate proportional to the product of the size of the two unengaged force levels. In this regard, the model has a certain similarity to the stochastic epidemic model.

There are three classifications; $X(t)$ and $Y(t)$, the unoccupied force levels at time t, and $Z(t)$, the engagements at time t. The $\{(X(t), Y(t), Z(t)), t \geq 0\}$ process can be modelled as a trivariate Markov process with transitions and rates given by

t		t + dt	Rate
(i,j,k)	→	(i+1,j,k-1)	$p_2\mu k dt + o(dt)$
		(i,j+1,k-1)	$p_1\mu k dt + o(dt)$
		(i+1,j+1,k-1)	$p_3\mu k dt + o(dt)$
		(i-1,j-1,k+1)	$\lambda N^{-1} ij dt + o(dt)$
		all others	$o(dt)$

(3.10)

where $N = X(0) + Y(0) + Z(0)$ and is assumed to be large. The approximate s.d.e.'s can be written

$$dX(t) = (p_2 + p_3)\mu Z(t)dt - \lambda N^{-1}X(t)Y(t)dt + (p_2\mu Z(t))^{1/2}dW_1(t)$$
$$+ (p_3\mu Z(t))^{1/2}dW_2(t) - (\lambda N^{-1}X(t)Y(t))^{1/2}dW_3(t)$$

$$dY(t) = (p_1 + p_3)\mu Z(t)dt - \lambda N^{-1}X(t)Y(t)dt + (p_1\mu Z(t))^{1/2}dW_4(t)$$
$$+ (p_3\mu Z(t))^{1/2}dW_2(t) - (\lambda N^{-1}X(t)Y(t))^{1/2}dW_3(t)$$

$$dZ(t) = -\mu Z(t)dt + \lambda N^{-1}X(t)Y(t)dt - (p_1\mu Z(t))^{1/2}dW_4(t)$$
$$- (p_2\mu Z(t))^{1/2}dW_1(t) - (p_3\mu Z(t))^{1/2}dW_3(t)$$
$$+ (\lambda N^{-1}X(t)Y(t))^{1/2}dW_3(t). \qquad (3.11)$$

Once again, we introduce the normalized process $\{(U_N(t), V_N(t), W_N(t)), t \geq 0\}$ where $U_N(t) = N^{-1/2}(X(t) - Nu(t))$, $V_N(t) = N^{-1/2}(Y(t) - Nv(t))$, and $W_N(t) = N^{-1/2}(Z_N(t) - Nw(t))$. This process will be a trivariate diffusion with s.d.e. which can be derived using Ito's Lemma. If one lets $N \to \infty$ and assumes u, v, and w satisfy

$$u'(t) = (p_2 + p_3)\mu w(t) - \lambda u(t)v(t)$$
$$v'(t) = (p_1 + p_3)\mu w(t) - \lambda u(t)v(t) \qquad (3.12)$$
$$w'(t) = -\mu w(t) + \lambda u(t)v(t),$$

then $\{(U_N(t), V_N(t), W_N(t)), t \geq 0\}$ converges weakly to $\{(U(t), V(t), W(t)), t \geq 0\}$ which satisfies

$$\begin{pmatrix} dU(t) \\ dV(t) \\ dW(t) \end{pmatrix} = \begin{pmatrix} -\lambda v(t) & -\lambda u(t) & (p_2 + p_3)\mu w(t) \\ -\lambda v(t) & -\lambda u(t) & (p_1 + p_3)\mu w(t) \\ \lambda v(t) & \lambda u(t) & -\mu w(t) \end{pmatrix} \begin{pmatrix} U(t) \\ V(t) \\ W(t) \end{pmatrix} dt \qquad (3.13)$$

(continued)

$$+\begin{pmatrix} (p_2\mu w(t))^{1/2} & 0 & (p_3\mu w(t))^{1/2} & -(\lambda u(t)v(t))^{1/2} \\ 0 & (p_1\mu w(t))^{1/2} & (p_3\mu w(t))^{1/2} & -(\lambda u(t)v(t))^{1/2} \\ -(p_2\mu w(t))^{1/2} & -(p_1\mu w(t))^{1/2} & -(p_3\mu w(t))^{1/2} & (\lambda u(t)v(t))^{1/2} \end{pmatrix}$$

$$\cdot \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \\ dW_4(t) \end{pmatrix},$$

or in matrix form

$$d\underset{\sim}{U}(t) = \underset{\sim}{A}_t\underset{\sim}{U}(t)dt + \underset{\sim}{B}_t d\underset{\sim}{W}(t) \qquad \text{with} \qquad \underset{\sim}{U}(0) = \underset{\sim}{0}.$$

The limiting stochastic process is linear, so the process will have trivariate normal marginals. The mean of $\underset{\sim}{U}(t)$ is $\underset{\sim}{0}$. The covariance at time t, $\underset{\sim}{\Sigma}_t$, is the unique non-negative definite solution of (2.9). Both equations (3.12), the classical Lanchester equations for this model, and (2.9) must be solved numerically. Since a numerical solution is required, the problem is essentially no harder to solve if λ, μ, and the p's are made explicitly time dependent.

The diffusion approximation thus characterizes the attrition process as being Gaussian with specified mean and covariance function, namely $N(u(t), v(t), w(t))$ and $N\underset{\sim}{\Sigma}_t$. The multivariate case thus poses no additional problems for this approximate methodology whereas the forward equations become more difficult to solve.

D. Heterogeneous Linear Law

Karr [10] has introduced a variety of different heterogeneous

models. These are characterized by each side having a number of different types of forces. Each force type on each side engages each force type of the other side. These engagements can be of any type; linear, quadratic, etc. Thus many different attrition processes are going on simultaneously. A rich variety of different models are possible, and we illustrate the diffusion approximation analysis in a few special cases. We present only the case where both sides have two distinct types of forces. The analysis can be easily generalized to the case where each side has many types. The reader should be aware, however, that while the mathematics is straightforward, the diffusion approximation approach is accurate only when the number of each type is reasonably large. As the number of types increases, this assumption will become doubtful.

We consider two force level process $X_1(t)$ and $X_2(t)$ for one side and $Y_1(t)$, $Y_2(t)$ for the other at time t. For a completely linear case, one would treat $\{(X_1(t), X_2(t), Y_1(t), Y_2(t)), t \geq 0\}$ as a continuous time Markov chain with transitions and transition rates given by

t		$t + dt$	Rate
(i,j,k,l)	$\rightarrow$	$(i-1,j,k,l)$	$i(c_1k + c_2l)dt + o(dt)$
		$(i,j-1,k,l)$	$j(d_1k + d_2l)dt + o(dt)$
		$(i,j,k-1,l)$	$k(e_1i + e_2j)dt + o(dt)$
		$(i,j,k,l-1)$	$l(f_1i + f_2j)dt + o(dt)$.

(3.14)

The diffusion approximation analysis would begin by approximating the process using s.d.e.'s given by

$$dX_1(t) = -X_1(t)(c_1N^{-1}Y_1(t) + c_2N^{-1}Y_2(t))dt$$
$$+ (c_1N^{-1}X_1(t)Y_1(t))^{1/2}dW_1(t) + (c_2N^{-1}X_1(t)Y_2(t))^{1/2}dW_2(t)$$

$$dX_2(t) = -X_2(t)(d_1N^{-1}Y_1(t) + d_2N^{-1}Y_2(t))dt$$
$$+ (d_1N^{-1}X_2(t)Y_1(t))^{1/2}dW_3(t) + (d_2N^{-1}X_2(t)Y_2(t))^{1/2}dW_4(t)$$

$$dY_1(t) = -Y_1(t)(e_1N^{-1}X_1(t) + e_2N^{-1}X_2(t))dt$$
$$+ (e_1N^{-1}X_1(t)Y_1(t))^{1/2}dW_5(t) + (e_2N^{-1}X_2(t)Y_1(t))^{1/2}dW_6(t)$$

$$dY_2(t) = -Y_2(t)(f_1N^{-1}X_1(t) + f_2N^{-1}X_2(t))dt$$
$$+ (f_1N^{-1}X_1(t)Y_2(t))^{1/2}dW_7(t) + (f_2N^{-1}X_2(t)Y_2(t))^{1/2}dW_8(t)$$

where $N = X_1(0) + X_2(0) + Y_1(0) + Y_2(0)$. (3.15)

The analysis proceeds in the same fashion as before. The normalized diffusion processes $U_{iN}(t) = (X_i(t) - Nu_i(t))/N^{1/2}$ and $V_{iN}(t) = (Y_i(t) - Nv_i(t))/N^{1/2}$, i=1,2 are introduced and the s.d.e. governing their behavior is derived using Ito's Lemma. Finally, one lets $N \to \infty$ assuming u_i and v_i, i=1,2 satisfy the ordinary differential equations

$$\begin{aligned} u_1'(t) &= -u_1(t)(c_1v_1(t) + c_2v_2(t)) \\ u_2'(t) &= -u_2(t)(d_1v_1(t) + d_2v_2(t)) \\ v_1'(t) &= -v_1(t)(e_1u_1(t) + e_2u_2(t)) \\ v_2'(t) &= -v_2(t)(f_1u_1(t) + f_2u_2(t)). \end{aligned} \tag{3.16}$$

The $\{(U_{1N}(t), U_{2N}(t), V_{1N}(t), V_{2N}(t)), t \geq 0\}$ process converges weakly to $\{(U_1(t), U_2(t), V_1(t), V_2(t)), t \geq 0\}$ which satisfies

$$\begin{pmatrix} dU_1(t) \\ dU_2(t) \\ dV_1(t) \\ dV_2(t) \end{pmatrix} = \begin{pmatrix} -(c_1v_1 + c_2v_2) & 0 & -c_1u_1 & -c_2u_1 \\ 0 & -(d_1v_1 + d_2v_2) & -d_1u_2 & -d_2u_2 \\ -e_1v_1 & -e_2v_1 & -(e_1u_1 + e_2u_2) & 0 \\ -f_1v_2 & -f_2v_2 & 0 & -(f_1u_1 + f_2u_2) \end{pmatrix} \begin{pmatrix} U_1(t) \\ U_2(t) \\ V_1(t) \\ V_2(t) \end{pmatrix} dt$$

$$+ \begin{pmatrix} (c_1u_1v_1)^{1/2} & (c_2u_1v_2)^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (d_1u_2v_1)^{1/2} & (d_2u_2v_2)^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (e_1u_1v_1)^{1/2} & (e_2u_2v_1)^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (f_1u_1v_2)^{1/2} & (f_2u_2v_2)^{1/2} \end{pmatrix} d\underset{\sim}{W}(t) \qquad (3.17)$$

where the t dependence of u_i and v_i has been supressed.

The analysis of (3.17) is straightforward, since it is linear. The distribution at time t will be multivariate normal with mean $\underset{\sim}{0}$ and covariance $\underset{\sim}{\Sigma}_t$ specified by (2.9). It follows that $(X_1(t), X_2(t), Y_1(t), Y_2(t))$ will have approximately a normal distribution with mean $N(u_1(t), u_2(t), v_1(t), v_2(t))$ and covariance $N\underset{\sim}{\Sigma}_t$. It appears that (3.16) and (2.9) must be solved numerically.

E. Heterogeneous Quadratic Law

The analysis presented in 3D can be easily modified to give an analysis of the heterogeneous quadratic case. Transitions given by (3.14) are replaced by

t		$t+dt$	rate
(i,j,k,l)	$\rightarrow$	$(i-1,j,k,l)$	$(c_1k + c_2l)dt + o(dt)$
		$(i,j-1,k,l)$	$(d_1k + d_2l)dt + o(dt)$
		$(i,j,k-1,l)$	$(e_1i + e_2j)dt + o(dt)$
		$(i,j,k,l-1)$	$(f_1i + f_2j)dt + o(dt)$

(3.18)

The deterministic equations given by (3.16) are replaced by

$$\begin{aligned} u_1'(t) &= -(c_1v_1(t) + c_2v_2(t)) \\ u_2'(t) &= -(d_1v_1(t) + d_2v_2(t)) \\ v_1'(t) &= -(e_1u_1(t) + e_2u_2(t)) \\ v_2'(t) &= -(f_1u_1(t) + f_2u_2(t)). \end{aligned} \tag{3.19}$$

These equations can be easily solved using standard methods. The functions u_1, u_2, v_1, and v_2 can be written as a sum of exponentials.

The stochastic noise term will satisfy a s.d.e. similar to (3.17). It will be given by

$$\begin{pmatrix} dU_1(t) \\ dU_2(t) \\ dV_1(t) \\ dV_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c_1 & -c_2 \\ 0 & 0 & -d_1 & -d_2 \\ -e_1 & -e_2 & 0 & 0 \\ -f_1 & -f_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1(t) \\ U_2(t) \\ V_1(t) \\ V_2(t) \end{pmatrix} dt \tag{3.20}$$

$$+ \begin{pmatrix} (c_1v_1)^{1/2} & (c_2v_2)^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (d_1v_1)^{1/2} & (d_2v_2)^{1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (e_1u_1)^{1/2} & (e_2u_2)^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (f_1u_1)^{1/2} & (f_2u_2)^{1/2} \end{pmatrix} d\underset{\sim}{W}(t)$$

where the t dependence of u_i and v_i, $i=1,2$ has been supressed.

The mean of $(U_1(t), U_2(t), V_1(t), V_2(t))$ is $\underset{\sim}{0}$ while the covariance matrix $\underset{\sim}{\Sigma}_t$ is the solution of (2.9). This matrix can be calculated in closed form and will be a mixture of exponentials.

The analysis presented in 3D and 3E can be easily generalized. Mixtures of attrition structures or indeed far more complicated structures can be easily accomodated. The latter two sections are meant only to illustrate the generalization to heterogeneous structures.

4. Numerical Results

Extensive numerical studies were carried out to assess the accuracy of the diffusion approximation methodology. The method is based on an asymptotic analysis where $X(0)$ and $Y(0)$ are assumed to be proportional to N and $N \to \infty$. Even so, we have claimed that the method provides good results for values of $X(0)$ and $Y(0)$ as small as 30. We show only two situations in this section, both based on the "quadratic law"; however, they are representative of the many cases considered.

A set of parameter values and starting values was chosen as well as a time T. The distribution is predicted by the diffusion approximation methodology to be bivariate normal with specified mean and covariance structure. To assess this prediction, the actual Markov chain given by (2.2) was simulated up to time T 6000 times. The simulation results were compared with the

approximation in several ways. First, the means, variances and covariances were compared. Second, the simulated distributions of X(T) and Y(T) were plotted on normal probability paper to determine the accuracy of the normal distribution approximation. Third, the (X(T), Y(T)) pairs were linearly transformed and the components plotted on normal probability paper to determine whether the joint distribution was really bivariate normal.

a = .05, b = .05, T = 15

	X(T) - mean			Y(T) - mean		
(x_0,y_0)	S	D	|S-D|/S	S	D	|S-D|/S
(20,20)	9.48 (.041)	9.45	.0032	9.40 (.002)	9.45	.0053
(25,25)	11.85 (.069)	11.81	.0034	11.87 (.061)	11.81	.0051
(30,30)	14.21 (.041)	14.17	.0028	14.16 (.011)	14.17	.0007
(40,40)	18.98 (.190)	18.89	.0047	18.93 (.069)	18.89	.0021
(50,50)	23.64 (.118)	23.62	.0008	23.58 (.251)	23.62	.0017

	X(T) - Standard Deviation			Y(T) - Standard Deviation		
(x_0,y_0)	S	D	|S-D|/S	S	D	|S-D|/S
(20,20)	3.87 (.053)	3.98	.0284	3.94 (.026)	3.98	.0102
(25,25)	4.42 (.050)	4.45	.0068	4.39 (.064)	4.45	.0137
(30,30)	4.84 (.004)	4.88	.0083	4.83 (.091)	4.88	.0104
(40,40)	5.66 (.136)	5.63	.0053	5.64 (.027)	5.63	.0018
(50,50)	6.27 (.031)	6.30	.0048	6.31 (.033)	6.30	.0016

(X(T), Y(T)) Correlation

(x_0,y_0)	S	D	|S-D|/S
(20,20)	-.694 (.005)	-.686	.0115
(25,25)	-.684 (.015)	-.686	.0029
(30,30)	-.683 (.001)	-.686	.0044
(40,40)	-.689 (.006)	-.686	.0044
(50,50)	-.687 (.009)	-.686	.0015

TABLE 1

a = .075, b = .030, T = 10

	X(T) - mean			Y(T) - mean		
(x_0,y_0)	S	D	|S-D|/S	S	D	|S-D|/S
(50,20)	40.18(.003)	40.16	.0005	6.78(.056)	6.72	.0088
(75,30)	60.32(.029)	60.24	.0013	10.08(.026)	10.09	.0010
(100,40)	80.34(.066)	80.32	.0002	13.45(.155)	13.45	.0000
(125,50)	100.32(.030)	100.41	.0009	16.86(.089)	16.81	.0030
(250,100)	200.96(.149)	200.81	.0007	33.70(.008)	33.62	.0024

	X(T) - Standard Deviation			Y(T) - Standard Deviation		
(x_0,y_0)	S	D	|S-D|/S	S	D	|S-D|/S
(50,20)	3.59(.021)	3.68	.0251	3.58 (.035)	3.84	.0726
(75,30)	4.43(.055)	4.51	.0181	4.61 (.074)	4.71	.0217
(100,40)	5.26(.049)	5.21	.0095	5.40 (.096)	5.43	.0056
(125,50)	5.80(.070)	5.82	.0034	6.08 (.088)	6.08	.0000
(250,100)	8.25(.238)	8.24	.0012	8.62 (.054)	8.59	.0035

(X(T), Y(T)) Correlation

(x_0,y_0)	S	D	|S-D|/S
(50,20)	-.512(.011)	-.526	.0273
(75,30)	-.534(.008)	-.526	.0150
(100,40)	-.547(.023)	-.526	.0384
(125,50)	-.518(.038)	-.526	.0154
(250,100)	-.527(.006)	-.526	.0019

TABLE 2

The columns marked S and D in Tables 1 and 2 represent the simulation and diffusion approximation results respectively. The quantity $|S-D|/S$ gives the percentage error. In the S column, for each starting value an estimate and its standard error are presented. Often the quantity $|S-D|$ is smaller than the standard error.

Tables 1 and 2 illustrate the extraordinary accuracy of the diffusion approximation method even for starting values as small as 20. The means are off by no more than 1/2 percent error, a quantity easily explained by the standard error of the simulation. Similarly an error of less than 3% is noted for the standard deviations and correlations and this drops to below 1/2 percent as the starting values increase. The diffusion approximation methodology predicts moments very accurately.

Figures 1, 2, and 3 give normal probability plots of X(T) and Y(T) for various parameter settings. The plots are shown for cases with a small starting values ((20,20) or (50,20)) and with a moderate starting values ((50,50) or (125,50)). The plots for small starting values exhibit some curvature indicative of non-normality; however, it is rather slight. This means that probabilities calculated from a normal distribution will be reasonably accurate.

The cases of moderate starting values show very strong normality (making allowances for sampling variability). These plots clearly indicate that when x_0 and y_0 are 50 or more excellent accuracy will be achieved using the diffusion approximation.

The plots of the transformed X(T) and Y(T) variables are not shown but gave results in accord with those illustrated in Figures 1, 2, and 3.

The figures also serve to illustrate the importance of using a stochastic analysis rather than a deterministic analysis. In Figure 1 with $(x_0, y_0) = (20,20)$, a deterministic analysis gives $(x(15), y(15)) = (9.45, 9.45)$. The graph shows $P(X(15) \leq 6) \cong .22$ and $P(X(15) \geq 12) \cong .22$, so $P(7 \leq X(15) \geq 11) \cong .56$. The deterministic value 9.45 is, therefore, quite misleading in view of the large variability. The case with $(x_0, y_0) = (50,50)$ is even more pronounced. Here $P(21 \leq X(T) \leq 29) \cong .52$, while the deterministic value is 23.62.

It is, therefore, clear that unless (x_0, y_0) is very large and the attrition essentially deterministic, a stochastic analysis must be employed. In view of the accuracy and convenience of the diffusion approximation it appears to be an ideal method.

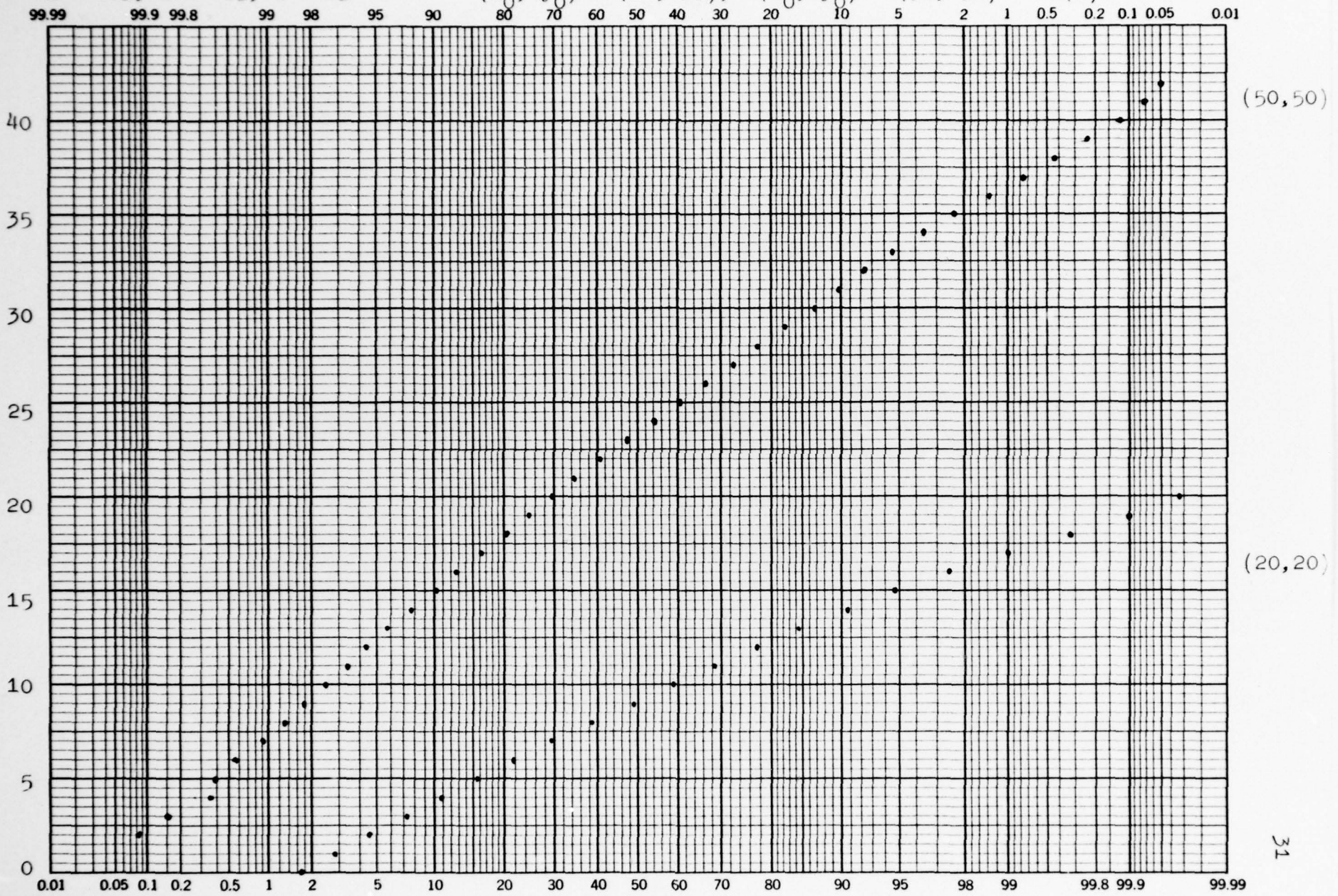


Figure 1

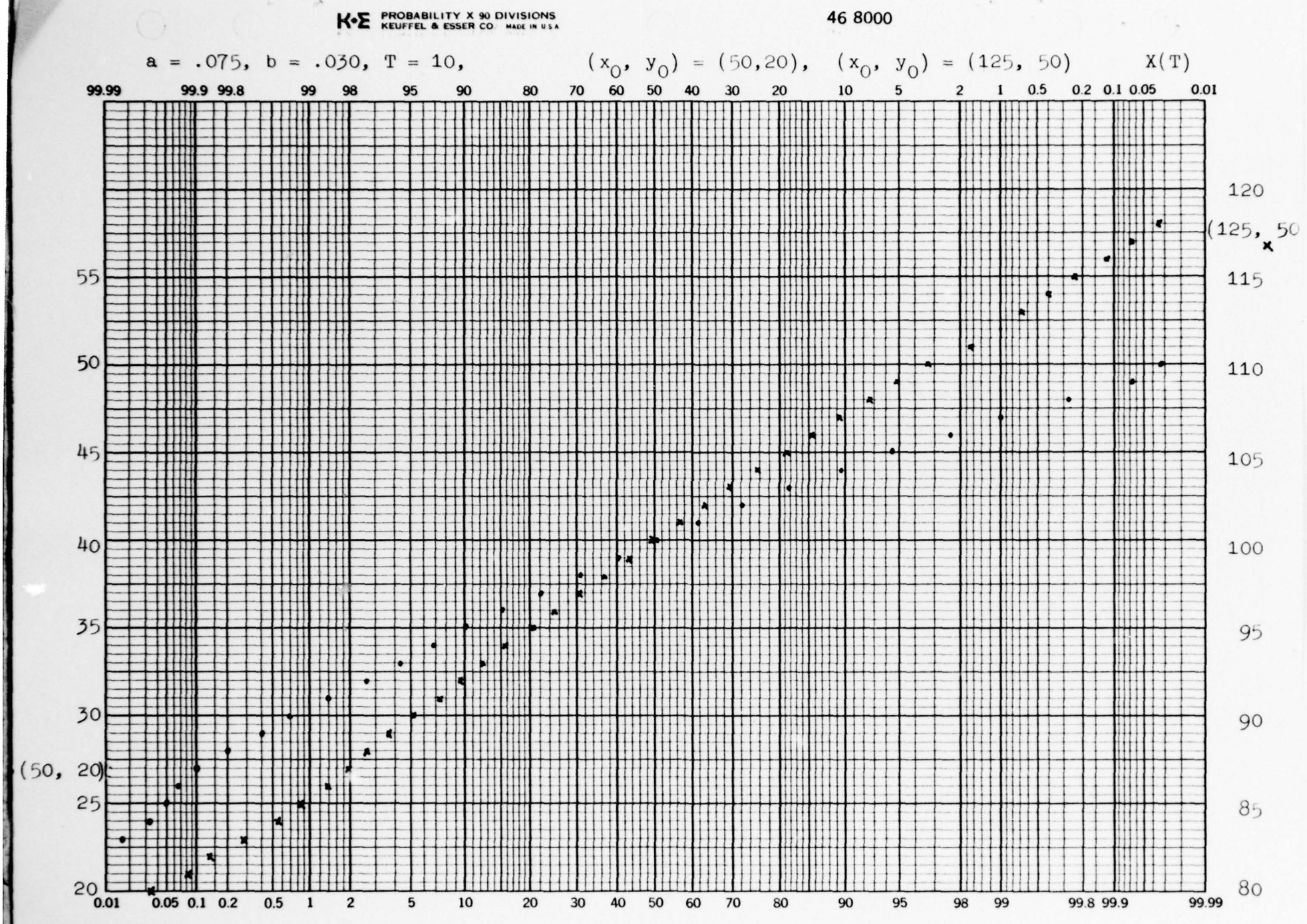
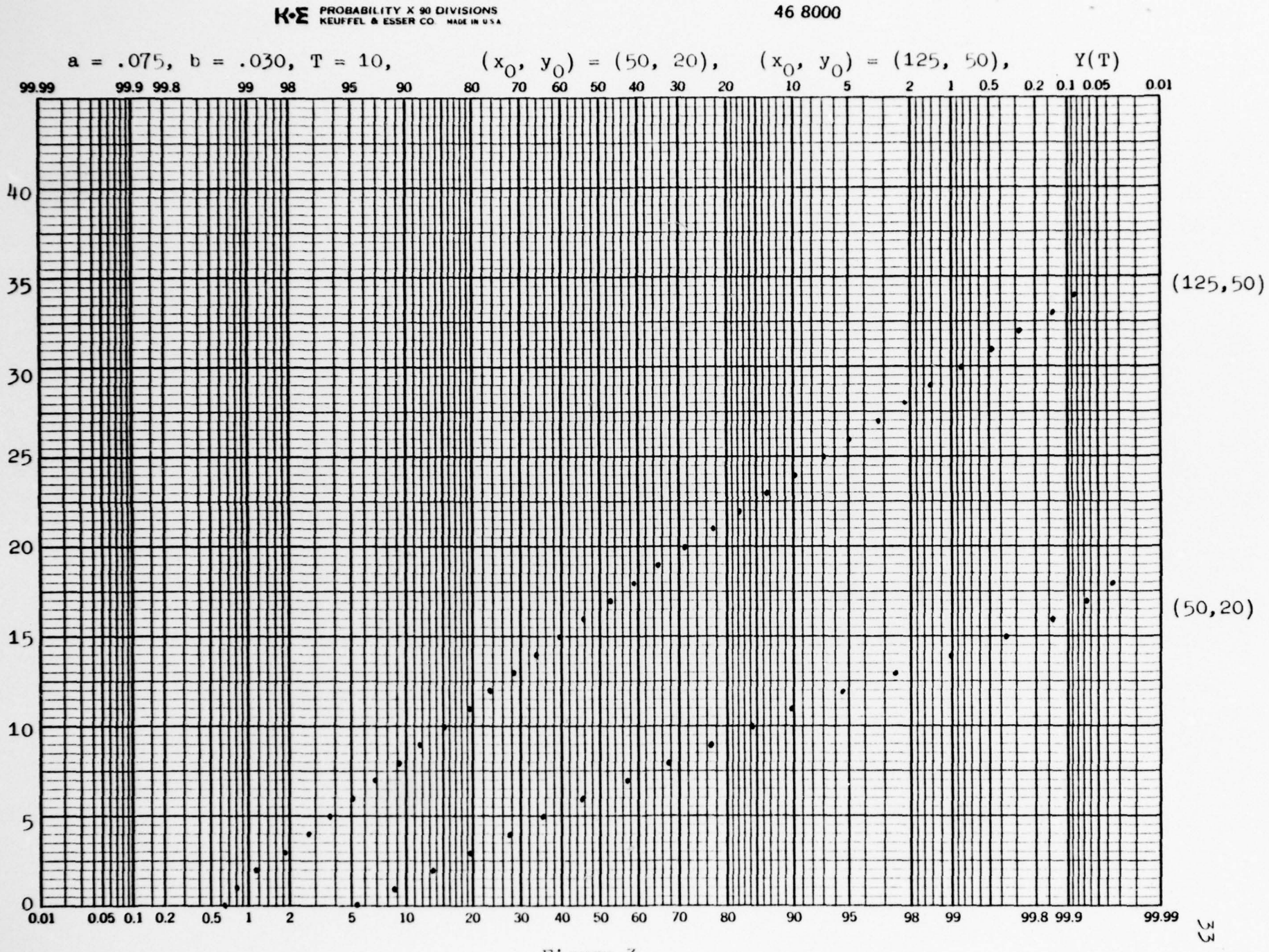

K•E PROBABILITY X 90 DIVISIONS
KEUFFEL & ESSER CO. MADE IN U.S.A.
46 8000
a = .075, b = .030, T = 10, $(x_0, y_0) = (50,20)$, $(x_0, y_0) = (125, 50)$ X(T)
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99
120 115 110 105 100 95 90 85 80
55 50 45 40 35 30 25 20
(125, 50)
(50, 20)

Figure 2


K·E PROBABILITY X 90 DIVISIONS
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46 8000
a = .075, b = .030, T = 10, $(x_0, y_0) = (50, 20)$, $(x_0, y_0) = (125, 50)$, Y(T)
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
40
35
30
25
20
15
10
5
0
(125,50)
(50,20)
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99

Figure 3

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